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Self-Dual Solutions for SU(2) and SU(3) Gauge Fields on Euclidean Space

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Self-dual solutions for SU(2) gauge fields on Euclidean space that satisfy Yang's ansatz are generalized by considering ρ as a function of ϕ for a special case when ρ is a complex analytic function and for SU(3) when $\bar{\rho_i}$, $i = 1, 2, 3$, are complex analytic functions.

KEY WORDS: gauge field theories; classical and semiclassical techniques; other nonperturbative techniques.

1. INTRODUCTION

To proceed in the study of self-dual SU(2) gauge field on Euclidean space we introduce the variables y, $\bar{y}$, z, $\bar{z}$ by the relations

$$\sqrt{2}y = x_1 + ix_2, \qquad \sqrt{2}\bar{y} = x_1 - ix_2,$$
$$\sqrt{2}z = x_3 + ix_4, \qquad \sqrt{2}\bar{z} = x_3 + ix_4. \tag{1.1}$$

The self-duality equations in four-dimensional Euclidean space $x_\mu = x_1, x_2, x_3, x_4$ are then (Yang, 1977)

$$F_{yz} = F_{\overline{yz}} = 0, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0, \tag{1.2}$$

where

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu - [A_\mu, A_\nu],$$
$$[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu \tag{1.3}$$

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is the curvature field taking its values in the algebra of SU(2) and SU(3) (antihermitian representation), and A_μ is the connection

$$A_y = D^{-1}D_y, \quad A_z = D^{-1}D_z, \quad A_{\bar{y}} = \bar{D}^{-1}\bar{D}_{\bar{y}}, \quad A_{\bar{z}} = \bar{D}^{-1}\bar{D}_{\bar{z}}, \tag{1.4}$$

where D and $\bar{D}$ are arbitrary 2×2 complex matrix functions of $y, \bar{y}, z, \bar{z}$ with determinant $= 1$ (for SU(2) gauge group) and $D_y = \partial_y D$ etc. For real gauge fields $A_\mu \doteq -A_\mu^+$ (the symbol $\doteq$ is used for equations valid only for real values of x_1, x_2, x_3, and x_4), we require

$$\bar{D} \doteq (D^+)^{-1}. \tag{1.5}$$

Gauge transformations are the transformations

$$D \to DU, \quad \bar{D} \to \bar{D}U, \quad U^+U \doteq I, \tag{1.6}$$

where U is a 2×2 complex matrix function of $y, \bar{y}, z$, and $\bar{z}$ with determinant $= 1$. Under transformation (1.6), Eq. (1.5) remains unchanged. We now define the hermitian matrix J as

$$J \equiv D\bar{D}^{-1} \doteq DD^+. \tag{1.7}$$

J has the very important property of being invariant under the gauge transformation equation (1.6). They only nonvanishing field strengths in terms of J become

$$F_{\mu\bar{v}} = -\bar{D}^{-1}(J^{-1}J_u)_{\bar{v}}\bar{D}, \tag{1.8}$$

$(u, v = y, z)$ and the remaining self-duality equation (1.2) takes the form

$$(J^{-1}J_y)_{\bar{y}} + (J^{-1}J_z)_{\bar{z}} = 0. \tag{1.9}$$

The action density in terms of J is

$$\begin{aligned}\Phi(J) &\equiv -\frac{1}{2}\mathrm{Tr}\, F_{\mu v}F_{\mu v} = -2\mathrm{Tr}(F_{y\bar{y}}F_{z\bar{z}} + F_{y\bar{z}}F_{\bar{y}z}) \\ &= -2\mathrm{Tr}\{(J^{-1}J_y)_{\bar{y}}(J^{-1}J_z)_{\bar{z}} - (J^{-1}J_y)_{\bar{z}}(J^{-1}J_z)_{\bar{y}}\}.\end{aligned} \tag{1.10}$$

The Atiyah–Ward (Atiyah and Ward, 1977) construction begins by an explicit parametrization of the matrix J

$$J = \begin{bmatrix} \dfrac{1}{\phi} & \dfrac{\bar{\rho}}{\phi} \\ \dfrac{\rho}{\phi} & \dfrac{\phi^2 + \rho\bar{\rho}}{\phi} \end{bmatrix}, \tag{1.11}$$

and for real gauge fields $A_\mu \doteq -A_\mu^+$ we require

$$\phi \doteq \text{real} \qquad \bar{\rho} \doteq \rho^* \quad (\rho^* \equiv \text{complex conjugate of}\, \rho). \tag{1.12}$$

The self-duality equations (1.9) take the form

$$\frac{1}{2}\Box \ln\phi + \frac{(\rho_y \bar{\rho}_{\bar{y}} + \rho_z \bar{\rho}_{\bar{z}})}{\phi^2} = 0, \tag{1.13}$$

$$\left(\frac{\rho_y}{\phi^2}\right)_{\bar{y}} + \left(\frac{\rho_z}{\phi^2}\right)_{\bar{z}} = 0, \tag{1.14}$$

$$\left(\frac{\bar{\rho}_{\bar{y}}}{\phi^2}\right)_{y} + \left(\frac{\bar{\rho}_{z}}{\phi^2}\right)_{z} = 0, \tag{1.15}$$

$$\Box = 2(\partial_y \partial_{\bar{y}} + \partial_z \partial_{\bar{z}}).$$

Using Eqs. (1.13)–(1.15) one can reduce the action density equation (1.10) to the following:

$$\Phi(\phi, \rho, \bar{\rho}) = -\frac{1}{2}\Box\Box \ln\phi + 2\left[\partial_y\partial_{\bar{y}}\left(\frac{\phi_z\phi_{\bar{z}} - \rho_y\bar{\rho}_{\bar{y}}}{\phi^2}\right) - \partial_y\partial_{\bar{z}}\left(\frac{\phi_z\phi_{\bar{y}} + \rho_z\bar{\rho}_{\bar{y}}}{\phi^2}\right)\right.$$
$$\left. + \partial_z\partial_{\bar{z}}\left(\frac{\phi_y\phi_{\bar{y}} - \rho_z\bar{\rho}_{\bar{z}}}{\phi^2}\right) - \partial_z\partial_{\bar{y}}\left(\frac{\phi_y\phi_{\bar{z}} + \rho_y\bar{\rho}_{\bar{z}}}{\phi^2}\right)\right]. \tag{1.16}$$

The Corrigan–Fairlie–‘t Hooft–Wilczek (CFTW) ansatz and ‘t Hooft’s solution have a simple form in terms of ϕ, ρ and $\bar{\rho}$.

CFTW ansatz

$$\rho_y = \phi_{\bar{z}}, \quad \rho_z = -\phi_{\bar{y}}, \quad \bar{\rho}_y = \phi_z, \quad \bar{\rho}_{\bar{z}} = -\phi_y,$$

‘t Hooft solution

$$\Box\phi = 0, \quad \phi = 1 + \sum_{j=1}^{q} \frac{\lambda_j^2}{(x - x_j)^2}, \tag{1.17}$$

$$\Phi(\phi, \rho, \bar{\rho}) = -\frac{1}{2}\Box\Box \ln\phi.$$

The paper is organized as follows. In Section 2 we give the reduced equations for self-dual SU(2) gauge fields giving new solutions with ρ as a function of ϕ, when ρ is a complex analytic function. In Section 3 the self-dual SU(3) Yang–Mills fields parametrised in a R-gauge are solved with a particular two function ansatz when $\bar{\rho}$ are complex analytic functions, $i = 1, 2, 3$.

2. ON THE YANG *R*-GAUGE FOR SU(2)

2.1. Self-Dual Solutions for SU(2) Gauge Fields on Euclidean Space When ρ Is a Function of ϕ

Yang (1977) has reduced the equations for self-dual SU(2) gauge fields on Euclidean space to the following equations

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \phi_y\phi_{\bar{y}} - \phi_z\phi_{\bar{z}} + \rho_y\bar{\rho}_{\bar{y}} + \rho_z\bar{\rho}_{\bar{z}} = 0,$$

$$\phi(\rho_{y\bar{y}} + \rho_{z\bar{z}}) - 2\rho_y\phi_{\bar{y}} - 2\rho_z\phi_{\bar{z}} = 0. \tag{2.1}$$

Let $\rho = \rho(\phi)$, then we find

$$\rho_y = \rho'\phi_y, \quad \rho_z = \rho'\phi_z, \quad \rho_{y\bar{y}} = \rho''\phi_y\phi_{\bar{y}} + \rho'\phi_{y\bar{y}}, \quad \text{and} \quad \rho_{z\bar{z}} = \rho''\phi_z\phi_{\bar{z}} + \rho'\phi_{z\bar{z}}.$$

Then Eq. (2.1) become

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) + (\rho'^2 - 1)(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0,$$

$$\phi\rho'(\phi_{y\bar{y}} + \phi_{z\bar{z}}) + (\phi\rho'' - 2\rho')(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0. \tag{2.2}$$

If we do not consider the case $\phi_{y\bar{y}} + \phi_{z\bar{z}} = 0$ and $\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}} = 0$ then we have

$$\phi\rho'' - \rho' - \rho'^3 = 0, \tag{2.3}$$

by integration we obtain

$$\rho' = \pm\frac{c\phi}{\sqrt{1 - c^2\phi^2}}, \tag{2.4}$$

and

$$\rho = \mp\frac{1}{c}\sqrt{1 - c^2\phi^2} + c'. \tag{2.5}$$

Equation (2.2) reduce to the Eq. (2.3). A solution is given by

$$\phi_y = \phi_z, \qquad \phi_{\bar{y}} = -\phi_{\bar{z}}. \tag{2.6}$$

A class of solutions is given by

$$\phi = F(y + z, \bar{y} - \bar{z}), \tag{2.7}$$

where F is an arbitrary function. Equations (2.5) and (2.7) gives a new set of solutions of Yang's equations for self-dual SU(2) gauge fields.

2.2. Self-Dual Solutions for SU(2) Gauge Fields on Euclidean Space When ρ Is a Complex Analytic Function

Since ρ is a complex analytic function of y and z, then we have

$$\rho_{\bar{y}} = \rho_{\bar{z}} = 0, \qquad \rho_{y\bar{y}} + \rho_{z\bar{z}} = 0. \tag{2.8}$$

The self-dual Yang–Mills equations take the form

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y \phi_{\bar{y}} + \phi_z \phi_{\bar{z}}) = 0, \qquad \rho_y \phi_{\bar{y}} + \rho_z \phi_{\bar{z}} = 0. \tag{2.9}$$

Yang (1977) has indicated the existence of a class of solutions of Eq. (2.9) that satisfies

$$\rho_y = \phi_{\bar{z}}, \qquad \rho_z = -\phi_{\bar{y}}, \tag{2.10}$$

yielding

$$\phi = \rho_y \bar{z} - \rho_z \bar{y} + f(y, z). \tag{2.11}$$

In the present section we seek to generalize the solutions by seeking solutions of Eq. (2.9) for the ansatz (Dipankar, 1980)

$$\rho_y = \psi_{\bar{z}}, \qquad \rho_z = -\psi_{\bar{y}}, \tag{2.12}$$

but ρ doesn't contain $\bar{y}$ and $\bar{z}$ and thus ψ may be almost linear in $\bar{y}$ and $\bar{z}$, where ψ is any complex function, from (2.12) we get the last equation of (2.8) and

$$\psi_{y\bar{y}} + \psi_{z\bar{z}} = 0. \tag{2.13}$$

Putting (2.12) into Eq. (2.9), gives

$$\psi = \psi(\phi). \tag{2.14}$$

Putting (2.14) into (2.13), gives

$$\psi_\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) + \psi_{\phi\phi}(\phi_y \phi_{\bar{y}} + \phi_z \phi_{\bar{z}}) = 0, \tag{2.15}$$

from Eq. (2.9) and (2.15) we obtain

$$\begin{vmatrix} \phi & -1 \\ \psi_\phi & \psi_{\phi\phi} \end{vmatrix} = 0, \qquad \frac{\psi''}{\psi'} = -\frac{1}{\phi},$$

then

$$\phi = c' e^{c\psi}, \tag{2.16}$$

where c and c' are integration constants.

Equations (2.11) and (2.16) gives a new set of solutions of Yang's equations for self-dual SU(2) gauge fields.

3. ON THE YANG *R*-GAUGE FOR SU(3) WHEN $\bar{\rho}_i$ ARE COMPLEX ANALYTIC FUNCTIONS, $i = 1, 2, 3$

The self-dual SU(3) Yang–Mills fields parametrised in a R-gauge has been formulated in Singh and Tchrakian (1981), Brihaye *et al.* (1978), and Prasad (1980). Singh and Tchrakian (1981) have achieved the integration of four of the eight SU(3) self-duality equations by virtue of the R-gauge parametrisation, and presented a particular two-function ansatz that solves all eight equations. In this section, following Singh and Tchrakian (1981), we achieve the integration of six of the eight SU(3) self-duality equations by the virtue of the R-gauge parametrisation. Further we present a particular two-function ansatz that solves all eight equations, by choosing $\bar{\rho}_i$ as complex analytic functions ($i = 1, 2, 3$). Now we solve when $\bar{\rho}_i$ are complex analytic function. At this point, we follow Yang (1977) to choose a gauge (the R-gauge) in parametrising the (3×3) unimodular matrices D and $\bar{D}$ as

$$D = R = (\phi_1\phi_2)^{-1/3}\begin{bmatrix} 1 & 0 & 0 \\ \rho_1 & \phi_1 & 0 \\ \rho_2 & \rho_3 & \phi_2 \end{bmatrix}, \tag{3.1}$$

$$\bar{D} = \bar{R} = (\phi_1\phi_2)^{1/3}\begin{bmatrix} 1 & -\bar{\rho}_1/\phi_1 & (\bar{\rho}_1\bar{\rho}_3 - \phi_1\bar{\rho}_2)/\phi_1\phi_2 \\ 0 & 1/\phi_1 & -\bar{\rho}_3/\phi_1\phi_2 \\ 0 & 0 & 1/\phi_2 \end{bmatrix}, \tag{3.2}$$

where, for real values of x_μ, $\bar{\rho}_i = \rho_i^*(i = 1, 2, 3)$, and $\bar{R} = (R^+)^{-1}$.

Using this parametrisation, the potentials take the form

$$A_y = \begin{bmatrix} \frac{-1}{3}\partial_y \ln \phi_1\phi_2 & 0 & 0 \\ \rho_{1y}/\phi_1 & \frac{-1}{3}\partial_y \ln\left(\phi_2/\phi_1^2\right) & 0 \\ \left[\rho_{2y} - \left(\frac{\rho_3}{\phi_1}\right)\rho_{1y}\right]\Big/\phi_2 & (\phi_1/\phi_2)\partial_y(\rho_3/\phi_1) & \frac{-1}{3}\partial_y \ln\left(\phi_1/\phi_2^2\right) \end{bmatrix},$$

$$A_{\bar{y}} = \begin{bmatrix} \frac{1}{3}\partial_{\bar{y}} \ln \phi_1\phi_2 & -\bar{\rho}_{1\bar{y}}/\phi_1 & -[\bar{\rho}_{2\bar{y}} - (\bar{\rho}_3/\phi_1)\bar{\rho}_{1\bar{y}}]/\phi_2 \\ 0 & \frac{1}{3}\partial_y \ln\left(\phi_2/\phi_1^2\right) & -(\phi_1/\phi_2)\partial_{\bar{y}}(\bar{\rho}_3/\phi_1) \\ 0 & 0 & \frac{1}{3}\partial_y \ln\left(\phi_1/\phi_2^2\right) \end{bmatrix}. \tag{3.3}$$

Before proceeding to investigate the remaining self-duality equation (1.2), we make a remark on the form of the gauge covariant hermitian matrix field

$$J = RR^+ = (\phi_1\phi_2)^{-2/3}\begin{bmatrix} 1 & \bar{\rho}_1 & \bar{\rho}_2 \\ \rho_1 & \rho_1\bar{\rho}_1 + \phi_1^2 & \rho_1\bar{\rho}_2 + \bar{\rho}_3\phi_1 \\ \rho_2 & \rho_2\bar{\rho}_1 + \rho_3\phi_1 & \rho_2\bar{\rho}_2 + \rho_3\bar{\rho}_3 + \phi_2^2 \end{bmatrix}. \tag{3.4}$$

Moreover the J matrix for SU(2) gauge fields in the R-gauge turned out to be related to the Atiyah–Ward ansatz (Atiyah and Ward, 1977).

There will be eight nonlinear differential equations involving the functions $\phi_1, \phi_2, \rho_i, \bar{\rho}_i (i = 1, 2, 3)$. To simplify these expressions we eliminate $(\rho_{1y}, \bar{\rho}_{1\bar{y}}, \rho_{1z}, \bar{\rho}_{1\bar{z}}), (\rho_{2y}, \bar{\rho}_{2\bar{y}}, \rho_{2z}, \bar{\rho}_{2\bar{z}})$, using the following replacements,

$$\bar{\Pi}_{\bar{z}} = (\rho_{2y} - \rho\rho_{1y})/\phi_2^2, \quad \bar{\Pi}_{\bar{y}} = -(\rho_{2z} - \rho\rho_{1z})/\phi_2^2, \tag{3.5}$$

$$\Pi_z = (\bar{\rho}_{2\bar{y}} - \overline{\rho\rho}_{1\bar{y}})/\phi_2^2, \quad \Pi_y = -(\bar{\rho}_{2\bar{z}} - \overline{\rho\rho}_{1\bar{z}})/\phi_2^2, \tag{3.6}$$

and

$$\bar{\psi}_{\bar{z}} = \left(\rho_{1y}/\phi_1^2\right) - \bar{\rho}\bar{\Pi}_{\bar{z}}, \quad \bar{\psi}_{\bar{y}} = -\left(\rho_{1z}/\phi_1^2\right) - \bar{\rho}\bar{\Pi}_{\bar{y}}, \tag{3.7}$$

$$\psi_z = \left(\bar{\rho}_{1\bar{y}}/\phi_1^2\right) - \rho\Pi_z, \quad \psi_y = -\left(\bar{\rho}_{1\bar{z}}/\phi_1^2\right) - \rho\Pi_y, \tag{3.8}$$

where $\rho = \rho_3/\phi_1$ and $\bar{\rho} = \bar{\rho}_3/\phi_1$. Then Eq. (1.2) can be expressed in the following form

$$\partial_y\Pi_z - \partial_z\Pi_y = 0, \quad \partial_{\bar{y}}\bar{\Pi}_{\bar{z}} - \partial_{\bar{z}}\bar{\Pi}_{\bar{y}} = 0, \tag{3.9}$$

$$\partial_y\psi_z - \partial_z\psi_y = 0, \quad \partial_{\bar{y}}\bar{\psi}_{\bar{z}} - \partial_{\bar{z}}\bar{\psi}_{\bar{y}} = 0, \tag{3.10}$$

$$\begin{aligned}
&(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\,\text{In}\,\phi_1 + \phi_1^2(\bar{\psi}_{\bar{z}}\psi_z - \bar{\psi}_{\bar{y}}\psi_y)\\
&\quad + \left(\phi_1^2\rho\bar{\rho} + 1/2\phi_2^2\right)(\bar{\Pi}_{\bar{z}}\Pi_z + \bar{\Pi}_{\bar{y}}\Pi_y)\\
&\quad + \phi_1^2\bar{\rho}(\bar{\Pi}_{\bar{z}}\psi_z + \bar{\Pi}_{\bar{y}}\psi_y) + \phi_1^2\rho(\Pi_z\bar{\psi}_{\bar{z}} + \Pi_y\bar{\psi}_{\bar{y}})\\
&\quad - \frac{1}{2}(\Phi_1/\phi_2)^2(\bar{\rho}_{\bar{z}}\rho_z + \bar{\rho}_{\bar{y}}\rho_y) = 0,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\,\text{In}\,\phi_2 + 1/2\phi_1^2(\bar{\psi}_{\bar{z}}\psi_z - \bar{\psi}_{\bar{y}}\psi_y)\\
&\quad + \left(1/2\phi_1^2\rho\bar{\rho} + \phi_2^2\right)(\bar{\Pi}_{\bar{z}}\Pi_z + \bar{\Pi}_{\bar{y}}\Pi_y)\\
&\quad + 1/2\phi_1^2\bar{\rho}(\bar{\Pi}_{\bar{z}}\psi_z + \bar{\Pi}_{\bar{y}}\psi_y) + 1/2\phi_1^2\rho(\Pi_z\bar{\psi}_{\bar{z}} + \Pi_y\bar{\psi}_{\bar{y}})\\
&\quad + \frac{1}{2}(\Phi_1/\phi_2)^2(\bar{\rho}_{\bar{z}}\rho_z + \bar{\rho}_{\bar{y}}\rho_y) = 0,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\rho + 2\rho_y\partial_{\bar{y}}\ln(\phi_1/\phi_2) + 2\rho_z\partial_{\bar{z}}\ln(\phi_1/\phi_2)\\
&\quad + \phi_2^2[(\Pi_{\bar{z}}\psi_z + \bar{\Pi}_{\bar{y}}\psi_y) + \rho(\bar{\Pi}_{\bar{z}}\Pi_z + \bar{\Pi}_{\bar{y}}\Pi_y)] = 0,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\bar{\rho} + 2\bar{\rho}_{\bar{y}}\partial_y\ln(\phi_1/\phi_2) + 2\bar{\rho}_{\bar{z}}\partial_z\ln(\phi_1/\phi_2)\\
&\quad + \phi_2^2[(\Pi_z\bar{\psi}_{\bar{z}} + \Pi_y\bar{\psi}_{\bar{y}}) + \bar{\rho}(\bar{\Pi}_{\bar{z}}\Pi_z + \bar{\Pi}_{\bar{y}}\Pi_y)] = 0.
\end{aligned} \tag{3.14}$$

Now we suppose that $\bar{\rho}_i$ are analytic function of y and z. Then Eq. (3.6) and (3.8) become

$$\Pi_y = \Pi_z = 0, \tag{3.15}$$

and

$$\psi_y = \psi_z = 0.$$

Then Eq. (1.2) can be expressed in the following form

$$\partial_{\bar{y}}\overline{\Pi}_{\bar{z}} - \partial_{\bar{z}}\overline{\Pi}_{\bar{y}} = 0, \tag{3.16}$$

$$\partial_{\bar{y}}\overline{\psi}_{\bar{z}} - \partial_{\bar{z}}\overline{\psi}_{\bar{y}} = 0, \tag{3.17}$$

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\ln\phi_1 - \frac{1}{2}(\phi_1/\phi_2)^2(\bar{\rho}_{\bar{z}}\rho_z + \bar{\rho}_{\bar{y}}\rho_y) = 0, \tag{3.18}$$

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\ln\phi_2 + \frac{1}{2}(\phi_1/\phi_2)^2(\bar{\rho}_{\bar{z}}\rho_z + \bar{\rho}_{\bar{y}}\rho_y) = 0, \tag{3.19}$$

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\rho + 2\rho_y\partial_{\bar{y}}\ln(\phi_1/\phi_2) + 2\rho_z\partial_{\bar{z}}\ln(\phi_1/\phi_2) = 0, \tag{3.20}$$

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\bar{\rho} + 2\bar{\rho}_{\bar{y}}\partial_y\ln(\phi_1/\phi_2) + 2\bar{\rho}_{\bar{z}}\partial_z\ln(\phi_1/\phi_2) = 0. \tag{3.21}$$

Thus we obtain six equations (3.16)–(3.21), Eqs. (3.16) and (3.17) are identically satisfied. This has been achieved by virtue of the replacements (3.15). There remains to solve the four equations (3.18)–(3.21), involving the four functions ϕ_1, ϕ_2, ρ, and $\bar{\rho}$. These are second-order coupled nonlinear equations. From Eqs. (3.18) and (3.19) we obtain

$$\phi_2 = \frac{1}{\phi_1}. \tag{3.22}$$

From Eqs. (3.18) and (3.22) we find

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\ln\phi_1 - \frac{1}{2}\phi_1^4(\bar{\rho}_{\bar{z}}\rho_z + \bar{\rho}_{\bar{y}}\rho_y) = 0. \tag{3.23}$$

From Eq. (3.20) we find

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\rho + 2\rho_y\partial_{\bar{y}}\ln\phi_1^2 + 2\rho_z\partial_{\bar{z}}\ln\phi_1^2 = 0. \tag{3.24}$$

Since ρ is a function of ϕ_1 then we obtain from (3.23) and (3.24)

$$\frac{1}{\phi_1}(\phi_{1y\bar{y}} + \phi_{1z\bar{z}}) + \left[\left(-1/\phi_1^2\right) - 1/2\left(\phi_1^4\rho'^2\right)\right](\phi_{1z}\phi_{1\bar{z}} + \phi_{1y}\phi_{1\bar{y}}) = 0, \tag{3.25}$$

$$\rho'(\phi_{1y\bar{y}} + \phi_{1z\bar{z}}) + [\rho'' + 4\rho'/\phi_1](\phi_{1z}\phi_{1\bar{z}} + \phi_{1y}\phi_{1\bar{y}}) = 0, \tag{3.26}$$

$$\rho'' + (5/\phi_1)\rho' + (1/2)\phi_1^5\rho'^3 = 0, \tag{3.27}$$

Consequently we obtain ρ_3

$$\rho_3 = \frac{i\phi_1\left(1 - 4c^2\phi_1^4\right)^{1/2}}{4c^2}\left[\phi_1^4 - \frac{1}{6}\left(1 - 4c^2\phi_1^4\right)\right] + c'\phi_1. \tag{3.28}$$

Similarly, from Eqs. (3.19) and (3.21) we obtain $\bar{\rho}_3$

$$\bar{\rho}_3 = \frac{-i\phi_1\left(1-4c^2\phi_1^4\right)^{1/2}}{4c^2}\left[\phi_1^4 - \frac{1}{6}\left(1-4c^2\phi_1^4\right)\right] + c'\phi_1,$$

where c and c' are integration constants.

From Eqs. (3.16) and (3.17) we can obtain ρ_{1y}, ρ_{1z}, ρ_{2y}, and ρ_{2z} as

$$\begin{aligned} \rho_{1y} &= \phi_{1\bar{z}}, \qquad \rho_{1z} = -\phi_{1\bar{y}}; \\ \rho_{2y} &= \phi_{2\bar{z}}, \qquad \rho_{2z} = -\phi_{2\bar{y}}. \end{aligned} \tag{3.29}$$

Equations (3.22), (3.28), and (3.29) gives a new set of solutions of Yang's equations for self-dual SU(3) gauge fields.

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